INVITATION TO INTER-UNIVERSAL TEICHMÜLLER THEORY (EXPANDED VERSION)

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"Travel and Lectures"

§1. **Hodge-Arakelov-theoretic Motivation**

For $l \geq 5$ a prime number, the module of l-torsion points associated to a **Tate curve** $E \stackrel{\text{def}}{=} \mathbb{G}_m/q^{\mathbb{Z}}$ (over, say, a p-adic field or $\mathbb{C}$) fits into a natural exact sequence:

$$0 \longrightarrow \boldsymbol{\mu}_l \longrightarrow E[l] \longrightarrow \mathbb{Z}/l\mathbb{Z} \longrightarrow 0$$

That is to say, one has canonical objects as follows:

a "**multiplicative subspace**" $\boldsymbol{\mu}_l \subseteq E[l]$ and "**generators**" $\pm 1 \in \mathbb{Z}/l\mathbb{Z}$

In the following, we fix an elliptic curve E over a number field F and a prime number $l \geq 5$. Also, we suppose that E has stable reduction at all finite places of F.

Then, in general, $E[l]$ does not admit

a **global** "**multiplicative subspace**" and "**generators**"

that coincide with the above canonical "multiplicative subspace" and "generators" at all finite places where E has bad mult. reduction! Nevertheless, let us suppose (!!) that such global objects do in fact exist. Then the **Fundamental Theorem** of **Hodge-Arakelov Theory** may be formulated as follows:

$$\Gamma(E^\dagger, \mathcal{L})^{<l} \quad \stackrel{\sim}{\to} \quad \bigoplus_{j=-l^*}^{l^*} \underline{\underline{q}}^{j^2} \cdot \mathcal{O}_F$$

— where

- $E^\dagger \to E$ is the "universal vectorial extension" of E;
- "$< l$" is the "relative degree" w.r.t. this extension; $l^* \stackrel{\text{def}}{=} (l-1)/2$;
- $\mathcal{L}$ is a line bundle that arises from a (nontrivial) 2-torsion point;
- "q" is the q-parameter at bad places of F; $\underline{\underline{q}} \stackrel{\text{def}}{=} q^{1/2l}$;
- LHS admits a **Hodge filtration** F^{-i} s.t. F^{-i}/F^{-i+1} is (roughly)

  $\stackrel{\sim}{\to} \ \omega_E^{\otimes(-i)} \qquad (i = 0, 1, \ldots, l-1;\ \omega_E = \text{cotang. bun. at the origin});$

- RHS admits a natural **Galois action** compatible with "$\bigoplus$".

This isom. is, *a priori*, only defined/F, but is in fact (essentially) compatible with the natural **integral structures**/**metrics** at all places of F.

A similar isom. may be considered over the moduli stack of elliptic curves. The proof of such an isom. is based on a computation, which shows that the **degrees $[-]$** of the vector bundles on either side of the isom. **coincide**:

$$\frac{1}{l} \cdot \text{LHS} \;\approx\; -\frac{1}{l} \cdot \sum_{i=0}^{l-1} i \cdot [\omega_E] \;\approx\; -\frac{l}{2} \cdot [\omega_E]$$

$$\frac{1}{l} \cdot \text{RHS} \;\approx\; -\frac{1}{l^2} \cdot \sum_{j=1}^{l^*} j^2 \cdot [\log(q)] \;\approx\; -\frac{l}{24} \cdot [\log(q)] \;=\; -\frac{l}{2} \cdot [\omega_E]$$

On the other hand, returning to the situation over number fields, since F^i is not compatible with the above direct sum decomposition, it follows that, by projecting to the factors of this direct sum decomp., one may construct a sort of relative of the so-called "arithmetic Kodaira-Spencer morphism", i.e., for (most) j, a (nonzero) morphism of line bundles

$$(\mathcal{O}_F \;\approx)\;\; F^0 \;\hookrightarrow\; \underline{\underline{q}}^{j^2} \cdot \mathcal{O}_F.$$

Since, moreover, $\deg_{\text{arith}}(F^0) \;\approx\; 0$, it follows that, if we denote the "height" determined by the log. diffs. $\Omega^{\log}_{\mathcal{M}}|_E$ associated to the moduli stack of elliptic curves by $\text{ht}_E \stackrel{\text{def}}{=} 2 \cdot \deg_{\text{arith}}(\omega_E) = \deg_{\text{arith}}(\Omega^{\log}_{\mathcal{M}}|_E)$, then we obtain an **inequality** (!) as follows:

$$\frac{1}{6} \cdot \deg_{\text{arith}}(\log(q)) \;=\; \text{ht}_E \;<\; \text{constant}$$

In fact, of course, since the global multiplicative subspace and generators which play an essential role in the above argument do **not**, in general, **exist**, this argument cannot be applied immediately in its present form.

This state of affairs motivates the following approach, which may appear somewhat **far-fetched** at first glance! Suppose that the assignment

$$\left\{\underline{\underline{q}}^{j^2}\right\}_{j=1,\dots,l^*} \quad\mapsto\quad \underline{\underline{q}}$$

somehow determines an **automorphism** of the **number field** F! Such an "automorphism" necessarily preserves degrees of arithmetic line bundles. Thus, since the absolute value of the degree of the RHS of the above assignment is "small" by comparison to the absolute value of the (average!) degree of the LHS, we thus conclude that a similar **inequality** (!) holds:

$$\frac{1}{6} \cdot \deg_{\text{arith}}(\log(q)) \;=\; \text{ht}_E \;<\; \text{constant}$$

Of course, such an automorphism of a NF does **not** in fact **exist**!! On the other hand, what happens if we regard the "$\{\underline{\underline{q}}^{j^2}\}$" on the LHS and the "$\underline{\underline{q}}$" on the RHS as belonging to **distinct** copies of "**conventional ring/scheme theory**" = "**arithmetic holomorphic structures**", and we think of the assignment under consideration

$$\left\{\underline{\underline{q}}^{j^2}\right\}_{j=1,\dots,l^*} \quad \mapsto \quad \underline{\underline{q}}$$

— i.e., which may be regarded as a sort of "**tautological solution**" to the

"obstruction to applying HA theory to diophantine geometry"

— as a sort of **quasiconformal map** between Riemann surfaces equipped with **distinct holomorphic structures**?

That is to say, this approach allows us to realize the assignment under consideration, albeit at the cost of **partially dismantling** conventional ring/sch. theory. On the other hand, this approach requires us

to compute just how much of a distortion occurs

as a result of dismantling = deforming conventional ring/scheme theory. This **vast computation** is the **content of IUTeich**.

In conclusion, at a concrete level, the "distortion" that occurs at the portion labeled by the index j is (roughly)

$$\leq \; j \cdot \text{log-diff}_F.$$

In particular, by the exact same computation (i.e., of the "leading term" of the average over j) as the computation discussed above in the case of the moduli stack of elliptic curves, we obtain the following **inequality**:

$$\tfrac{1}{6} \cdot \deg_{\text{arith}}(\log(q)) \;=\; \text{ht}_E \;\leq\; (1+\epsilon)\text{log-diff}_F \;+\; \text{constant}$$

This inequality is the content of the so-called

Szpiro Conjecture ($\Longleftrightarrow$ **ABC Conjecture**).

§2. <u>**Teichmüller-theoretic Deformations**</u>

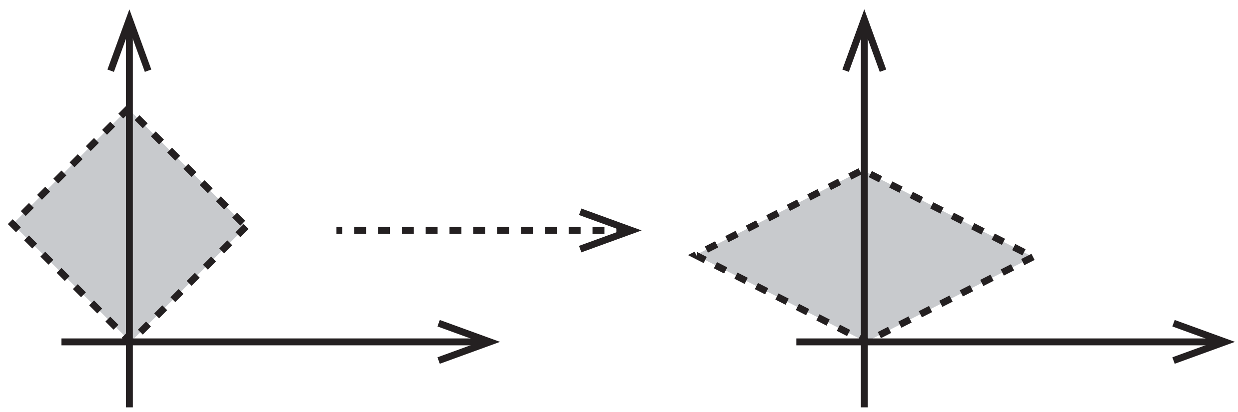
<u>Classical Teichmüller theory over $\mathbb{C}$</u>:

Relative to the canonical coordinate $z = x + iy$ (associated to a square differential) on the Riemann surface, **<u>Teichmüller deformations</u>** are given by

$$z \mapsto \zeta = \xi + i\eta = Kx + iy$$

— where $1 < K < \infty$ is the <u>dilation</u> factor.

<u>Key point</u>: **<u>one</u>** holomorphic dim., but **<u>two</u>** underlying real dims., of which **<u>one</u>** is **<u>dilated/deformed</u>**, while the **<u>other</u>** is left **<u>fixed/undeformed</u>**!



<u>*p*-adic Teichmüller theory</u>:

· **<u>*p*-adic canonical liftings</u>** of a hyperbolic curve in positive characteristic equipped with a nilpotent indigenous bundle

· **<u>canonical Frobenius liftings</u>** over the ordinary locus of the moduli stack of curves, as well as over tautological curve — cf. the metric on the **<u>Poincaré</u>** upper half-plane, **<u>Weil-Petersson metric</u>** in the theory/$\mathbb{C}$.

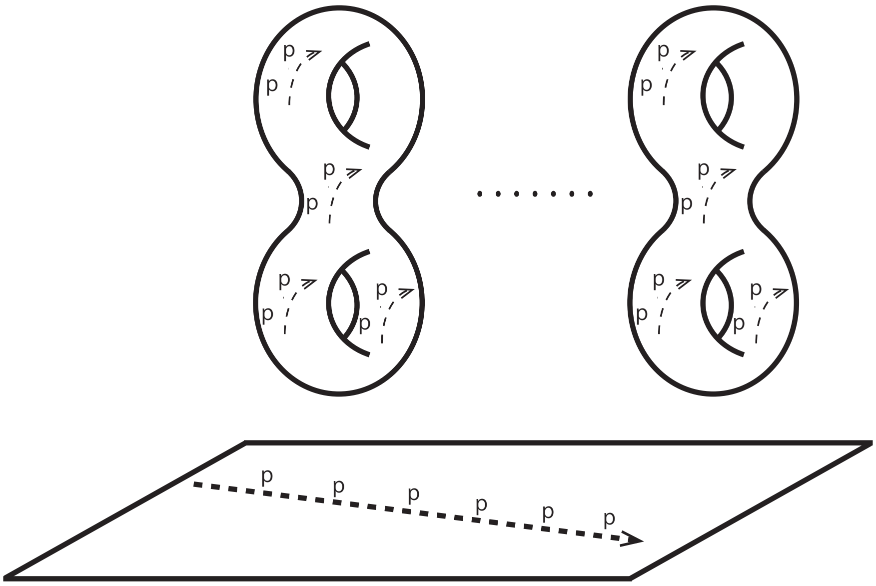
<u>Analogy between IUTeich and *p*Teich</u>:

conventional scheme theory/$\mathbb{Z}$ $\longleftrightarrow$ scheme theory/$\mathbb{F}_p$

number field (+ fin. many places) $\longleftrightarrow$ hyperbolic curve in pos. char.

once-punctured elliptic curve/NF $\longleftrightarrow$ nilpotent indigenous bundle

<u>log-Θ-lattice</u> $\longleftrightarrow$ **<u>*p*-adic canonical lifting</u>** + **<u>canonical Frob. lifting</u>**



The arithmetic case: addition and multiplication, cohom. dim.:

Regard the **ring structure** of rings such as $\mathbb{Z}$ as a

one-dimensional "arithmetic holomorphic structure"!

— which has

two underlying combinatorial dimensions!

"addition"	and	"multiplication"
$(\mathbb{Z}, +)$	$\curvearrowleft$	$(\mathbb{Z}, \times)$
one combinatorial dim.		one combinatorial dim.

— cf. the **two cohomological dims.** of the absolute Galois group of

- a (totally imaginary) number field $F/\mathbb{Q} < \infty$,
- a p-adic local field $k/\mathbb{Q}_p < \infty$,

(Note: the pro-l-related portion of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is $\approx \mathbb{Z}_l \rtimes \mathbb{Z}_l^{\times}$),

as well as the **two underlying real dims.** of

- $\mathbb{C}^{\times}$.

Units and value groups:

In case of a p-adic local field $k/\mathbb{Q}_p < \infty$, one may also think of these **two underlying combinatorial dimensions** as follows:

$$\mathcal{O}_k^\times \quad \subseteq \quad k^\times \quad \twoheadrightarrow \quad k^\times/\mathcal{O}_k^\times \; (\cong \mathbb{Z})$$

one combinatorial dim. one combinatorial dim.

— cf. the direct product decomp. in the complex case: $\mathbb{C}^\times = \mathbb{S}^1 \times \mathbb{R}_{>0}$.

In IUTeich, we shall **deform the holom. str. of the number field** by **dilating the value groups** via the **theta function**, while leaving the **units undilated**!

§3. The Log-theta-lattice

Noncommutative (!) 2-dim. diagram of Hodge theaters "•":

2 dims. of the diagram $\longleftrightarrow$ **2 comb. dims.** of a p-adic local field!

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
\dots & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow & & \\
\dots & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & &
\end{array}$$

Analogy between IUTeich and pTeich:

$\bullet$ = a copy of scheme theory/$\mathbb{Z}$ $\longleftrightarrow$ a copy of scheme theory/$\mathbb{F}_p$

$\uparrow$ = $\mathfrak{log}$-link $\longleftrightarrow$ the Frob. morphism in pos. char.

$\longrightarrow$ = Θ-link $\longleftrightarrow$ $\left(p^n/p^{n+1} \rightsquigarrow p^{n+1}/p^{n+2}\right)$

[$\Theta^{\pm \mathrm{ell}}$NF-]Hodge theaters:

A "[$\Theta^{\pm \mathrm{ell}}$NF-]Hodge theater" is a model of the **conventional scheme-theoretic arithmetic geometry** surrounding an elliptic curve E over a number field F. At a more concrete level, it is a complicated system of

abstract monoids and **Galois groups/arith. fund. gps.**

that arise naturally from E/F and its various localizations.

The **principle** that underlies this system: the system serves as
a **bookkeeping apparatus** for the ***l*-tors. points** that allows one
to simulate a **global** **multiplicative subspace** + **generators** (cf. §1)!

$$\rightsquigarrow \ \mathbb{F}_l^{\divideontimes}\text{-}, \ \mathbb{F}_l^{\rtimes \pm}\text{-symmetries}$$

$$(\text{where} \quad \mathbb{F}_l^{\divideontimes} \overset{\mathrm{def}}{=} \mathbb{F}_l^{\times}/\{\pm 1\}, \quad \mathbb{F}_l^{\rtimes \pm} \overset{\mathrm{def}}{=} \mathbb{F}_l \rtimes \{\pm 1\})$$

$$\overset{\{\pm 1\}}{\curvearrowright} \quad \begin{pmatrix} -l^* < \dots < -1 < 0 \\ < 1 < \dots < l^* \end{pmatrix} \ \Rightarrow \ \begin{bmatrix} 1 < \dots \\ < l^* \end{bmatrix} \ \Leftarrow \ \begin{pmatrix} 1 < \dots \\ < l^* \end{pmatrix}$$

$$\Downarrow \qquad\qquad\qquad\qquad\qquad\qquad \Downarrow$$

$$\begin{array}{ccc} \pm & \rightarrow & \pm \\ \uparrow & \overset{\mathbb{F}_l^{\rtimes \pm}}{\curvearrowright} & \downarrow \\ \pm & \leftarrow & \pm \end{array} \qquad\qquad\qquad\qquad \begin{array}{ccc} \divideontimes & \rightarrow & \divideontimes \\ \uparrow & \overset{\mathbb{F}_l^{\divideontimes}}{\curvearrowright} & \downarrow \\ \divideontimes & \leftarrow & \divideontimes \end{array}$$

Hint that underlies the construction of this apparatus: **global** **mult. s/sp.** on the moduli stack of elliptic curves over $\mathbb{Q}_p$, as in ***p*-adic Hodge theory**

(p-adic Tate module) $\otimes$ (p-adic ring of fns.)

... $\rightsquigarrow$ "**combinatorial rearrangement**" of **basepoints** by means of a **mysterious** '$\otimes$'!

$\longleftrightarrow$ the **absolute anabelian geometry** applied in a Hodge theater!

$\mathfrak{log}$-Link:

At nonarchimedean v of the number field F, the ring structures on either side of the $\mathfrak{log}$-link are related by a **non-ring-homomorphism** (!)

$$\log_v : \overline{k}^{\times} \;\to\; \overline{k}$$

— where $\overline{k}$ is an algebraic closure of $k \stackrel{\mathrm{def}}{=} F_v$; $G_v \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$.

Key point: The $\mathfrak{log}$-link is **compatible** with the isomorphism

$$\Pi_v \;\stackrel{\sim}{\to}\; \Pi_v$$

between the arithmetic fundamental groups Π_v on either side of the $\mathfrak{log}$-link, relative to the **natural actions** via $\Pi_v \twoheadrightarrow G_v$. Moreover, if one allows v to vary, the $\mathfrak{log}$-link is also compatible with the action of the global absolute Galois groups. Finally, at archimedean v of F, one has an analogous theory.

Θ-Link:

At bad nonarchimedean v of the number field F, the ring structures on either side of the Θ-link are related by a **non-ring-homomorphism** (!)

$$\mathcal{O}^{\times}_{\overline{k}} \;\stackrel{\sim}{\to}\; \mathcal{O}^{\times}_{\overline{k}}; \qquad \Theta|_{l\text{-tors}} = \left\{ \underline{\underline{q}}^{j^2} \right\}_{j=1,\dots,l^*} \;\mapsto\; \underline{\underline{q}}$$

— where $\overline{k}$ is an algebraic closure of $k \stackrel{\mathrm{def}}{=} F_v$; $G_v \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$.

Key point: The Θ-link is **compatible** with the isomorphism

$$G_v \;\stackrel{\sim}{\to}\; G_v$$

between the Galois groups G_v on either side of the Θ-link, relative to the **natural actions** on $\mathcal{O}^{\times}_{\overline{k}}$. At good nonarchimedean/archimedean v of F, one can give an analogous definition, by applying the **product formula**.

Remark: It is only possible to define the "**walls/barriers**" (i.e., from the point of view of the **ring structure** of conventional ring/scheme theory) constituted by the $\mathfrak{log}$-, Θ-links by working with

abstract monoids/...

— i.e., of the sort that appear in a Hodge theater!

Remark: By contrast, the objects that appear in the **étale-picture** (cf. the diagram below!) — i.e., the portion of the log-theta-lattice constituted by the

arithmetic fundamental groups/Galois groups

— have the power to

slip through these "walls"!

Various versions of "**Kummer theory**" — which allow us to relate the following two types of mathematical objects:

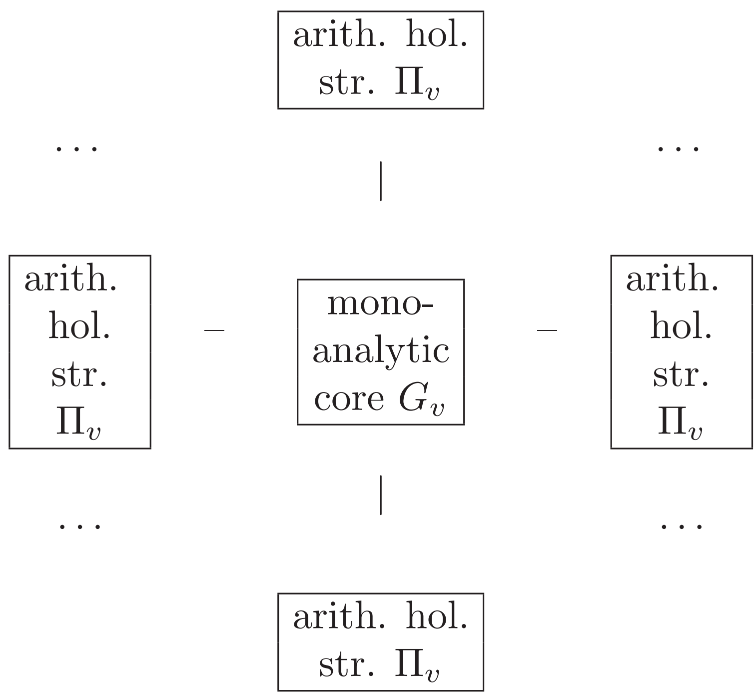
abstract monoids = **Frobenius-like** objects and
arith. fund. gps./Galois groups = **étale-like** objects

— play a very important role throughout IUTeich! Moreover, the transition

「**Frobenius-like** ⇝ **étale-like**」

may be regarded as a **global analogue over number fields** of the computation — i.e., via "**cartesian coords.** ⇝ **polar coords.**" — of the classical **Gaussian integral**

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \;\; !$$



Main objects to which **Kummer theory** is **applied** (cf. **LHS** of **Θ-link**!):

(a) **gp. of units** $\mathcal{O}^{\times}_{\overline{k}} \curvearrowleft \widehat{\mathbb{Z}}^{\times}$ (nonarch. v)

(b) **values of theta fn.** $\Theta|_{l\text{-tors}} = \left\{\underline{\underline{q}}^{j^2}\right\}_{j=1,\dots,l^*}$ (bad nonarch. v)

(c) a sort of "**realification**" of the **number field** F

Main focus of the theory is to protect the **cyclotomes** ($\cong \widehat{\mathbb{Z}}(1)$) contained in the monoids where (b), (c) appear from the **indeterminacy** "$\curvearrowleft \widehat{\mathbb{Z}}^{\times}$", i.e., **cyclotomic rigidity**!

Case of (b): **theory of étale theta fn.** $\Longrightarrow$ cyclo. rig.
Case of (c): **elem. alg. no. theory** $\Longrightarrow$ cyclo. rig.

The Kummer theory of (b), (c) is well-suited to the resp. portions of a Hodge theater where the **symmetries** act (cf. the chart below)!

This state of affairs closely resembles the (well-known) elementary theory of the "**functions**" associated to the various **symmetries** of the classical upper half-plane $\mathfrak{H}$ (cf. the chart below)!

	The classical upper half-plane $\mathfrak{H}$	$\Theta^{\pm\text{ell}}$**NF-Hodge theaters in IUTch**
(Cuspidal) add. symm.	$z \mapsto z + a$, $z \mapsto -\overline{z} + a$ $(a \in \mathbb{R})$	$\mathbb{F}_l^{\rtimes\pm}$-**symmetry**
"**Functions**" assoc. to add. symm.	$q \stackrel{\text{def}}{=} e^{2\pi i z}$	$\Theta\|_{l\text{-tors}} = \left\{\underline{\underline{q}}^{j^2}\right\}_{j=1,\dots,l^*}$
(Nodal/toral) mult. symm.	$z \mapsto \frac{z\cdot\cos(t)-\sin(t)}{z\cdot\sin(t)+\cos(t)}$, $z \mapsto \frac{\overline{z}\cdot\cos(t)+\sin(t)}{\overline{z}\cdot\sin(t)-\cos(t)}$ $(t \in \mathbb{R})$	$\mathbb{F}_l^{*}$-**symmetry**
"**Functions**" assoc. to mult. symm.	$w \stackrel{\text{def}}{=} \frac{z-i}{z+i}$	elements of **no. fld.** F

In fact, this portion of IUTeich closely resembles, in many respects (cf. the chart below!), **Jacobi's identity**

$$\theta(t) \;=\; t^{-1/2} \cdot \theta(1/t)$$

— which may be thought of as a sort of **function-theoretic** version of the **Gaussian integral** that appeared in the discussion above — concerning the classical **theta function** on the upper half-plane

$$\theta(t) \;=\; \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}.$$

IUTeich	**Theory of Jacobi's identity**
rigidity properties of étale theta fn.	**invariance** of **Gaussian distrib.** w.r.t. **Fourier transform**
the **indeterminacy** $\mathcal{O}^{\times}_{\overline{k}} \curvearrowleft \widehat{\mathbb{Z}}^{\times}$	unit factor in Fourier transform $\int(-)\cdot e^{it}$, $t \in \mathbb{R}$
proof of rig. properties via **quad'icity** of theta gp. $[-,-]$	proof of Fourier invariance via **quad'icity** of exp. of Gauss. dist.
$\left\{\underline{\underline{q}}^{j^2}\right\}_{j=1,\dots,l^*}$	**Gaussian** expansion of theta fn.
Abs. anab. geom. applied to **rotation** of $\boxplus$, $\boxtimes$ via $\mathfrak{log}$-link	**Analytic continuation** $\infty \rightsquigarrow 0$, the **rotation** $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \Longleftrightarrow t \mapsto \frac{1}{t}$
Local/global functoriality of abs. anab. algorithms, **Belyi cuspidalization**	**Zariski-localizable isomorphism** betw. **Galois cohom.** and **diffs.** in p-adic Hodge theory

§4. **Inter-universality and Anabelian Geometry**

Note that the $\mathfrak{log}$-, Θ-links are **not compatible** with the **ring structures**

$$\log_v : \overline{k}^{\times} \ \to \ \overline{k}, \qquad\qquad \Theta|_{l\text{-tors}} = \left\{ \underline{\underline{q}}^{j^2} \right\}_{j=1,\dots,l^*} \ \mapsto \ \underline{\underline{q}}$$

in their domains and codomains, hence are **not compatible**, in a quite **essential** way, with the **scheme-theoretic** "**basepoints**" and

$$\textbf{Galois groups} \qquad (\quad \subseteq \mathrm{Aut}_{\mathrm{field}}(\overline{k}) \ \ !! \quad)$$

that arise from ring homomorphisms! That is to say, when one passes to the "opposite side" of the $\mathfrak{log}$-, Θ-links,

"Π_v" and "G_v"

only make sense in their capacity as **abstract topological groups**!

$\Longrightarrow$ As a consequence, in order to compute the relationship between the ring structures in the domain and codomain of the $\mathfrak{log}$-, Θ-links, it is necessary to apply **anabelian geometry**! At the level of previous papers by the author, we derive the following Main Theorem by applying the results and theory of

- · Semi-graphs of Anabelioids
- · The Geometry of Frobenioids I, II
- · The Étale Theta Function ...
- · Topics in Absolute Anab. Geo. III

concerning

absolute anabelian geometry and

various rigidity properties of the étale theta function.

Main Theorem: One can give an **explicit, algorithmic description**, up to mild indeterminacies, of the **LHS** of the **Θ-link** in terms of the "**alien**" **ring structure** on the **RHS** of the Θ-link.

Key points:

· the **coricity** (i.e., coric nature) of $G_v \curvearrowright \mathcal{O}^{\times}_{\overline{k}}$!

· various versions of "**Kummer theory**", which allow us to relate the following two types of mathematical objects (cf. the latter portion of §3):

abstract monoids = **Frobenius-like** objects and
arith. fund. gps./Galois groups = **étale-like** objects.

Here, we recall the analogy with the computation of the **Gaussian integral**:

definition of **$\mathfrak{log}$-, Θ-link**, log-theta-lattice ⟷ **cartesian coords.**

algor. descr. via **abs. anab. geom.** ⟷ **polar coords.**

crucial rigidity of **cyclotomes** ($\cong \widehat{\mathbb{Z}}(1)$) ⟷ coord. trans. via $\mathbb{S}^1 \curvearrowright$

· The **$\mathfrak{log}$-link** plays an indispensable role in the context of realizing the action on the "**log-shell**" = "**container**"

$$\log(\mathcal{O}^{\times}_{\overline{k}}) \curvearrowleft \left\{\underline{\underline{q}}^{j^2}\right\}_{j=1,\dots,l^*}$$

... but various technical difficulties arise as a consequence of the **noncommutativity** of the **log-theta-lattice**.

⟹ in the subsequent "volume computation", one only obtains an **inequality** (i.e., not an equality)!

By performing a **volume computation**, as discussed in §1, concerning the output of the algorithms of the above Main Theorem, one obtains:

Corollary: The "**Szpiro Conjecture**" (⟺ "**ABC Conjecture**").

This portion of the theory resembles, in many respects, the theory surrounding **Jacobi's identity**, as discussed at the end of §3:

IUTeich	**Theory of Jacobi's identity**
Changes of **universe**, i.e., **labeling apparatus** for **sets**	Changes of **coordinates**, i.e., **labeling apparatus** for **points** of a **space**
computation of volume of **log-shell** $\log(-)$	computation via **polar coordinates** of **Gaussian integral** $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$
Startling application to **diophantine geometry**	**Startling** improvement in **computational accuracy** of values of classical theta function

Relative to the analogy with the classical theory concerning hyperbolic curves over p-adic local fields and the geometry of Riemann surfaces over $\mathbb{C}$, the corresponding **inequalities** (which may be regarded as expressions of "**hyperbolicity**") are as follows:

- the degree $= (2g-2)(1-p) \leq 0$ of the

  $$\text{"}\textbf{Hasse invariant} = \frac{1}{p} \cdot d(\textbf{Frob. lift.})\text{"}$$

  in ***p*Teich**,
- the **Gauss-Bonnet Theorem** for a hyperbolic Riemann surface S

  $$0 \ > \ -\int_S (\text{Poincaré metric}) \ = \ 4\pi(1-g).$$